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Coupled coincidence fixed point theorems in partially ordered metric spaces which endowed with vector-valued metrics

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ABSTRACT: In this paper, the existence and uniqueness of coupled xed point for mapping having the mixed monotone property in partially ordered Banach spaces which endowed with vector-valued metrics and some result in C*-algebras are given.

Keywords: Fixed points, Complete generalized metric space, Fixed points

INTRODUCTION

Let X be a nonempty set. A mapping d : X X ! R^m is called a vector-valued metric on X if the following properties are satis ed:

(1) d(x; y) 0 for each x; y 2 X; if d(x; y) = 0, then x = y;
(2) d(x; y) = d(y; x) for each x; y 2 X;
(3) d(x; y) d(x; z) + d(z; y) for each x; y; z 2 X.

A set X equipped with a vector-valued metric d is called a generalized metric space and denoted by (X; d). By $M_{m;m}(R^+)$ we mean that the set of all m m matrices with positive elements. We denote by the zero matrix, and by **I** the identity m m matrix. Let A 2 $M_{m;m}(R^+)$, A is said to be convergent to zero if and only if A^n ! 0 as n ! 1 (for more details see [7]).

Let ; 2 R^m, = ($_1$; $_2$; ; $_n$), = ($_1$; $_2$; ; $_n$), and c 2 R. By (resp. <) we mean that $_i$ $_i$ (resp. $_i$ < $_i$) for each 1 i m, and by c (resp. < c) for 1 i m.
Notice that for the proof of the main results, we need the following equivalent statements

(1) A is convergent towards zero;
(2) A^n ! 0 as n ! 1;
(3) The eigenvalues of A are in the open uint disc, that is, j j < 1, for each 2 C with det(A **I**) = 0;
(4) The matrix **I** A is nonsingular and
(**I** A) ^{1} = **I** + A + + A^n + ;
(5) A^nq ! 0 and qA^n ! 0 as n ! 1, for each q 2 R^m.

Where the proof of the above statements are the classical results in matrix analysis (for more details see [1], [5], and [6]).

De nition 1.1 ([3]). Let (X;) be a partially ordered set and F : X X ! X. Mapping F is said to be has the mixed monotone property if F (x; y) is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for every x; y 2 X,

(i) for each x_1; x_2 2 X, if x_1 x_2, then F (x_1; y) F (x_2; y);
(ii) for each y_1; y_2 2 X, if y_1 y_2, then F (x_1; y) F (x_2; y).

Let (X;) be a partially ordered set and d be a metric on X such that (X; d) is a complete metric space. The product space X X is endowed with the following partial order:

for (x; y); (u; v) 2 X X; (u; v) (x; y) , x u; y v:

De nition 1.2 ([3]). Let (X;) be a partially ordered set and F : X X ! X. An element (x; y) 2 X X is said to be a coupled xed point of the mapping F , if F (x; y) = x and F (y; x) = y.

Gnana Bhaskar and Lakshmikantham in [3], proved the following important Theorem:

Theorem 1.3. [3, Theorem 2.1]Let (X;) be a partially ordered set and suppose that there exists a metric d on X such that (X; d) is a complete metric space. Let F : X X ! X be a continuous mapping having the mixed monotone property on X. Assume that there exists a k 2 [0; 1) with

d(F (x; y); F (u; v)) $\frac{k}{2}$ [d(x; u); d(y; v)];

for all x u and y v. If there exist two elements x_0; y_0 2 X with

x_0 F (x_0; y_0) and y_0 F (x_0; y_0):

Then there exist x; y 2 X such that

x = F (x; y) and y = F (y; x):

De nition 1.4. An element (x; y) 2 X X is called

(1) a coupled coincidence point of mappings F : X X ! X and g : X ! X if g(x) = F (x; y) and g(y) = F (y; x), and (gx; gy) is called a coupled point of coincidence.

(2) a common coupled xed point of mappings F : X X ! X and g : X ! X if x = g(x) = F (x; y) and y = g(y) = F (y; x).

De nition 1.5. (Let (X;) be a partially ordered set and F : X X ! X and g : X ! X be two self mappings. F has the mixed g-monotone property if F is monotone g-non-decreasing in its rst argument and is monotone g-non-increasing in its second argument, that is, if for all x_1; x_2 2 X, gx_1 gx_2 implies F (x_1; y) F (x_2; y) for any y 2 X, and for all y_1; y_2 2 X, gy_1 gy_2 implies F (x; y_1) F (x; y_2) for any x 2 X.

De nition 1.6. Let X be a non-empty set. We say that the mappings F : X X ! X and g : X ! X are commutative if g(F (x; y)) = F (gx; gy), for all x; y 2 X.

Main Results

Theorem 2.1. Let (X;) be partial ordered Banach space, and F : X X ! X and g : X ! X . and F mapping having the mixed g monotone property on X. Assume that there exists A 2 Mm m(R+); A 6= I be a nonzero matrix converging to zero whit:

(2.1) jj(F (x; y) F (u; v))jj A[jjgx gujj + jjgy gvjj];

for all x; y; u; v 2 X for which g(x) g(u) and g(v) g(y). Suppose that F (X X) g(X), g is sequentially continuous and commutes with F and also suppose either F is continuous or X has the following property:

(I) if a non-decreasing fxng ! x, then xn x, for all n.

(II) if a non-decreasing fyng ! y, then y yn, for all n.

If there exist x0; y0 2 X such that g(x0) F (x0; y0) and g(y0) such that g(x) = F (x; y) and g(y) = F (y; x); that is, F and g have

Proof. Let x_0; y_0 2 X with g(x_0) F (x_0; y_0) = x_1 and g(y_0) F (y_0; x_0) = g(x_1). Suppose that g(x_2) = F (x_1; y_1) and g(y_2) = F (y_1; x_1). Continuing this process, we have g(x_{n+1}) = F (x_n; y_n) and F (y_n; x_n) = g(x_{n+1}) for all n 0: Thus g(x_n) g(x_{n+1}; and g(y_{n+1}): Therefore the g-monotone property of F implies

g(x_{n+1}) = F (x_n; y_n) F (x_n; y_n); and F (y_n; x_n) = g(y_{n+1}):

Thus F (x_{n+1}; y_n) F (x_{n+1}; y_{n+1}) = g(x_{n+2}), g(y_{n+2}) = F (y_{n+1}; x_{n+1}) F (y_{n+1}; x_n):

Then we have g(x_{n+1}) g(x_{n+2}) and g(y_{n+2}) g(y_{n+1}). Therefore

g(x_0) g(x_1) g(x_2) g(x_n) g(x_{n+1}) ;

and

$g(y_0)$ $g(y_1)$ $g(y_2)$ $g(y_n)$ $g(y_{n+1})$:

We show that sequences fg(x_n)g and fg(y_n)g are Cauchy:

jjg(x_n) g(x_{n+1})jj = jjF ($x_{n\ 1}$; $y_{n\ 1}$) F (x_n; y_n)jj

A

2 [jjg($x_{n\ 1}$) g(x_n)jj + jjg($y_{n\ 1}$) g(y_n)jj]

A^2

2 [jjg($x_{n\ 2}$) g($x_{n\ 1}$)jj + jjg($y_{n\ 2}$) g($y_{n\ 1}$)jj]

$_A$n

(2.2) 2 [jjg(x_0) g(x_1)jj + jjg(y_0) g(y_1)jj]:

Similarly

jjg(y_n) g(y_{n+1})jj = jjF ($y_{n\ 1}$; $x_{n\ 1}$) F (y_n; x_n)jj

A[jjg($y_{n\ 1}$) g(y_n)jj jjg($x_{n\ 1}$) g(x_n)jj]

A

2 [jjg($x_{n\ 1}$) g(x_n)jj + jjg($y_{n\ 1}$) g(y_n)jj]

A^2

2 [jjg($y_{n\ 2}$) g($y_{n\ 1}$)jj + jjg($x_{n\ 2}$) g($x_{n\ 1}$)jj]

A^n

(2.3) 2 [jjg(y_0) g(y_1)jj + jjg(x_0) g(x_1)jj]:

Together with (2.2) and (2.3) we have

jjg(x_n) g(x_{n+1})jj + jjg(y_n) g(y_{n+1})jj A^n[jjg(x_0) g(x_1)jj + jjg(y_0) g(y_1)jj]:

For n > m, we have

jjg(x_n) g(x_m)jj + jjg(y_n) g(y_m)jj

jjg(x_n) g($x_{n\ 1}$)jj + jjg(y_n) g($y_{n\ 1}$)jj + ::: + jjg(x_m) g(x_{m+1})jj

+jjg(x_m) g(x_{m+1})jj + jjg(y_m) g(y_{m+1})jj

($A^{n\ 1}$ + $A^{n\ 2}$ + :::A^m)[jjg(x_0) g(x_1)jj + jjg(y_0) g(y_1)jj]

(2.4) A^m(I A) ^{1}[jjg(x_0) g(x_1)jj + jjg(y_0) g(y_1)jj:

Thus sequences fg(x_n)g and fg(y_n)g are Cauchy

Since X is Banach algebra then these sequence are convergence. Thus there exists x; y 2 X such that

lim g(x_n) = x; and lim g(y_n) = y:
n!1 n!1

By continuity of g, $\lim_{n!1}$ g(g(x_{n+1}) = g(x) and $\lim_{n!1}$ g(g(y_{n+1}) = g(y), and by commutativity of F and g, we have

g(g(x_{n+1}) = g(F (x_n; y_n)) = F (g(x_n); g(y_n));

and

g(g(y_{n+1})) = g(F (y_n; x_n)) = F (g(y_n); g(x_n)):

Now we show that F (x; y) = g(x) and F (y; x) = g(y):

Frits case: Let F be continuous.

g(x) = lim g(g(x_{n+1})) = lim F (g(x_n); g(y_n)) = F (lim g(x_n); lim g(y_n)) = F (x; y);
n!1 n!1 n!1 n!1

and

g(y) = lim g(g(y_{n+1})) = lim F (g(y_n); g(x_n)) = F (lim g(y_n); lim g(x_n)) = F (y; x):
n!1 n!1 n!1 n!1

Second case: Now, suppose that (I) and (II) hold. Since g(x_n) ! x and g(y_n) ! y; then by (I) and (II), g(x_n) x and y g(y_n) for all n. Thus

jjg(x) F (x; y)jj		jjg(x) g(g(x_{n+1}))jj + jjg(g(x_{n+1})) F (x; y)jj	
	=	jjg(x) g(g(x_{n+1}))jj + jjF (g(x_n); g(y_n)) F (x; y)jj	
			A
(2.5)	jjg(x) g(g(x_{n+1}))jj +	$\overline{2}$	[jjg(g(x_n)) g(x)jj + jjg(g(y_n) g(y)jj]:

Hence, take the limit of both sides as n ! 1; we have jjg(x) F (x; y)jj 0: Thus g(x) = F (x; y) and

similarly g(y) = F (y; x):

Theorem 2.2. Under the hypothesis of Theorem 2.1, suppose that for every (x; y); (x^0; y^0) 2 X X;

there exists a couple (u; v) 2 X X such that (F (u; v); F (v; u)) and (F (x^0; y^0); F (y^0; x^0)): Then F and g have a unique couple common xed point, in other word, there exists a unique (x; y) 2 X X such that x = g(x) = F (x; y), and y = g(y) = F (y; x).

Proof. Existence of the set of coupled coincidence points is due to theorem 2.1. Let (x; y); (x^0; y^0) 2 X X; be the coupled coincidence points, that is g(x) = F (x; y); g(y) = F (y; x) and g(x^0) = F (x^0; y^0); F (y^0; x^0) = g(y^0): By assumption, there is a couple (u; v) 2 X X such that (F (u; v); F (v; u)) is comparable to (F (x; y); F (y; x)) and (F (x^0; y^0); F (y^0; x^0)): Set u_0 = u; v_0 = v and choose u_1; v_1 2 X with g(u_1) =
F (u_0; v_0); g(v_1) = F (v_0; u_0):

Similar to the proof of theorem 2.1, we construct the sequences fg(u_n)g and fg(v_n)g in the way that
g(x_{n+1} = F (u_n; v_n); g(v_{n+1} = F (v_n; u_n). Similarly we can construct the the sequences fg(x_n)g, fg(y_n)g; fg(x^0_n)g and fg(y_n^0)g:

x_0 = x) g(x_{n+1}) = F (x_n; y_n);
y_0 = y) g(y_{n+1}) = F (y_n; x_n);
x^0_0 = x^0) g(x^0_{n+1}) = F (x^0_n; y_n^0);
and
y_0^0 = y^0) g(y_{n+1}^0) = F (y_n^0; x^0_n):

Since (g(x); g(y)) = (F (x; y); F (y; x)) = (g(x_1); g(y_1)) and (F (u; v); F (v; u)) = (g(u_1); g(v_1)) are com-parable, then g(x) g(u_1) and g(v_1) g(y): Similarly (g(x); g(y)) and (g(u_1); g(v_1)) are comparable, that is g(x) g(u_n) and g(v_n) g(y), for n 1,

A
jjg(x) g(u_{n+1})jj = jjF (x; y) F (u_n; v_n)jj $\overline{2}$ [jjg(x) g(u_n)jj + jjg(y) g(v_n)jj];
and
A
jjg(y) g(v_{n+1})jj = jjF (y; x) F (v_n; u_n)jj $\overline{2}$ [jjg(y) g(v_n)jj + jjg(x) g(u_n)jj]:
Which imply that
jjg(x) g(u_{n+1})jj + jjg(y) g(v_{n+1})jj A[jjg(x) g(u_n)jj + jjg(y) g(v_n)jj]:
Thus
jjg(x) g(u_{n+1})jj + jjg(y) g(v_{n+1})jj A^n[jjg(x) g(u_1)jj + jjg(y) g(v_1)jj]:
If n ! 1 then A^n ! 0, then jjg(x) g(u_{n+1})jj + jjg(y) g(v_{n+1})jj ! 0. Therefore

$\lim_{n!1}$ jjg(x) g(u_{n+1})jj = 0; and $\lim_{n!1}$ jjg(y) g(v_{n+1})jj = 0:

Similarly

$\lim_{n!1}$ jjg(x^0) g(u_{n+1})jj = 0; and $\lim_{n!1}$ jjg(y^0) g(v_{n+1})jj = 0:

Thus
jjg(x) g(x^0)jj jjg(x) g(u_{n+1})jj + jjg(u_{n+1}) g(x^0)jj ! 0 as n ! 1;
jjg(y) g(y^0)jj jjg(y) g(v_{n+1})jj + jjg(v_{n+1}) g(y^0)jj ! 0 as n ! 1:
Therefore g(x) = g(x^0) and g(y) = g(y^0). By of commutativity of F and g with g(x) = F (x; y) and g(y) = F (y; x), we get g(g(x)) = g(F (x; y)) = F (g(x); g(y));
and

g(g(y)) = g(F (y; x)) = F (g(y); g(x)):

By letting t = g(x) and s = g(y), then g(t) = F (t; s) and g(s) = F (s; t). This means that (t; s) is coupled coincidence point, also g(x) = g(t) and g(y) = g(s), where $t = x^0$ and $s = y^0$.
Since t = g(x) and s = g(y), then g(t) = t and g(s) = s. So (t; s) is coupled common xed point of F and g.
Uniqueness, follows form $g(x) = g(x^0)$ and $g(y) = g(y^0)$. Indeed, for another coupled common xed

point (t; s) of F and g, then t = g(t) = g(t) = t and s = g(s) = g(s) = s:

We now present some results in C -algebras.

Theorem 2.3. Let A be a unital C -algebra and let F : A A A A 2 ! A 2 be a holomorphic map that satis es the conditions F (0; 0) = 0, $\frac{@F}{@x}$ (0; 0) = id A, $\frac{@F}{@y}$ (0; 0) = 0, $\frac{@F}{@x^2}$ (0; 0) = 0, $\frac{@F}{@y^2}$ (0; 0) = 0, and $\frac{@^2F}{@y@x}$ (0; 0) = 0. Then every (a; b) 2 A A \ Z(A A) is a coupled xed point for F . Furthermore (a ; b) is a coupled xed point of F .

Proof. Since every unital C -algebra is semisimple ([8, Corollary 3.2.13]), so by Theorem 3.1 of [9], every (a; b) 2 A A \ Z(A A) is a coupled xed point for F . Now, suppose x = (a; b); $y = (a^0; b^0)$ 2 A A. Since jjxjj = jjx jj, therefore if x 2 A A T $Z_{A A}$ then x 2 A A T $Z_{A A}$. As well as, (x y) = y x = xy = (yx) that is x y = yx .

Theorem 2.4. Let A be a unital C -algebra, let F : A A A A ! A, and let g : A ! A such that F has the mixed g-monotone property. Assume g is biholomorphic function from A into A such that g(0) = 0 and $g^0(0) = id_A$. Then g is -preserving on A T Z_A.

Proof. Let x 2 A T Z_A, by theorem (2.3) x 2 A T Z_A and theorem of [10], g(A T Z_A) = A T Z_A, we have g(x) = x = g(x)

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